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The image shows a screenshot of a Google Scholar search for "python in scientific computing". A green circle highlights the top search result: "Python for scientific computing" by J. Oliphant, published in *Computing in Science & Engineering*, 2007. The snippet below the title reads: "By itself, Python is an excellent scripting language for scientific computing. However, with additional basic tools, Python transforms into a language suited for scientific and engineering code that's often faster than C. Cited by 690 Related articles All 12 versions Cite Save". To the right of the search results is a book cover for "Computing in Science & Engineering" featuring a colorful illustration of a city and a globe, with "NERSC" and "National Energy Research Scientific Computing Center" visible. On the left side of the screenshot, there is a green vertical banner with the text "Searching? Trust CiSE." and on the right, another green banner that says "It's peer-reviewed and appears in the IEEE Xplore and AIP library packages."

Two-space, two-time similarity solution for decaying homogeneous turbulence

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A two-point, two-time similarity solution is derived for homogeneous decaying turbulence. This is the first known solution which includes the temporal decay at two-different times. It assumes that the turbulence is homogeneous in all three space dimensions, and finds that homogeneity holds across time. The solutions show that time is logarithmically “stretched” while the homogeneous spatial scales grow. This solution reduces to the two point, single time equation when the two times are set equal. The turbulence initially decays exponentially, then asymptotically as t^{-n} where $n \geq 1$ and equality is possible only if the initial energy is infinite. The methodology should be applicable to other non-equilibrium homogeneous turbulent flows. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4974355>]

I. INTRODUCTION

It has been customary since the analysis of De Karman and Howarth¹ to treat homogeneous decaying turbulence by examining only the spatial correlation (or their spectral counterparts) as a function of time. For the most part experiments have used decaying turbulence in wind-tunnels together with Taylor’s frozen field hypothesis to interpret the results.^{2–5} Numerical studies of both forced and decaying turbulence have followed the same path, treating time as an independent variable only coupled through time-dependent coefficients to the spatial correlations and spectra.

There has never been completely satisfactory agreement between the results of either simulations or experiments and theory. The non-locality and time dependence of the turbulent equations of motion lead to a difficulty in producing a model of the flow. One aspect of this difficulty lies in the modeling of different turbulence quantities, namely, the diffusion, production, and dissipation terms of the TKE equation. Efforts have included a turbulent time scale as the second variable in two-equation turbulence modeling to successfully replicate empirical results in the effort to simplify the calculation of flow fields.⁶ Additionally, models that incorporated multiple independently calculated time scales to better capture the turbulence acting across a spectrum of scales showed promising results.⁷

George⁸ showed that part of the problem with the earlier similarity solutions was that they were over-constrained. The revised equilibrium similarity approach of George fixed some of the problems, but even so, discrepancies (e.g., the integral scale and the derivative skewness) remained problematic, especially relative to the experiments. Some of the questions could be rationalized by short-comings of the experiments in modeling the assumed theoretical conditions, or vice-versa.

The most obvious difference is the finite scale of an experiment since a homogeneous theory is by definition infinite in extent. Another challenge with fixed and bounded domains in both experiments and simulations (at least to-date) is the fact that the relevant turbulence scales changed with time. As a consequence, the spatial resolution changed with time, since the computational grid (or tunnel size) was fixed while the scales grew. While there will be sufficient resolution of the small scales of the flow, a decreased domain size relative to the large scales can result in the boundaries affecting the turbulence and leading to a departure from theories based on an infinite domain. There have been multiple instances of a dynamically scaled model that can instantaneously calculate the sub-grid coefficients for large eddy simulations.^{9,10} These methodologies are able to more accurately follow experimental and DNS calculations than previous single scale models, and the use of homogeneous and isotropic theory to develop the dynamic model shows the importance of understanding these small scales.

Another potential problem (particularly with simulations in Fourier space) was the triadic interactions that should have occurred with neglected wavenumbers representing large and smaller scales. Energy which should have left the chosen wavenumber domain was “trapped” in the modes chosen, with a result that energy piled up at both the highest and lowest wavenumbers. So the DNS was at best a periodic turbulence, and only a valid approximation to a homogeneous theory for only a limited time, and even then for not all scales.

As noted by George,¹¹ turbulence (at least of the homogeneous variety) is a four-dimensional phenomenon in which all modes and time are coupled. This is easily demonstrated for forced (or statistically stationary) turbulence by decomposing it in both time and space. The non-linear terms lead to convolutions in both wavenumber and frequency. So the triadic interactions are four-dimensional (three component of wavenumber plus frequency), not three. And there is no direct correspondence between the Fourier coefficients of this

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four-dimensional decomposition, say $\hat{u}_i(\mathbf{k}, \omega)$, and the results of the usual three-dimensional decomposition, say $\tilde{u}_i(\mathbf{k}, t)$. Thus frequency information is smeared across the wavenumber domain in the latter representation, and the resulting Fourier coefficients may not accurately represent the temporal behavior since all phase information is lost. This problem does not seem to have been previously noted, but it is potentially significant. Even if the turbulence is decaying and temporal Fourier modes are not an appropriate decomposition, any other decomposition will yield similar results—four-dimensional convolutions where no simple time dependence is possible.

The problem of how to decompose any flow was addressed by Lumley,¹² who looked for deterministic solutions by maximizing the projection of an unknown function onto the random velocity field. He showed that almost any flow could be decomposed optimally in terms of maximizing the energy in each mode if the two-point, two-time four-dimensional space-time correlations were known. For flows of finite total energy, this result was the well-known POD, and the solutions were empirical eigenfunctions given by a four-dimensional integral equation with the two-point Reynolds stress tensor as kernel. For infinite dimension, however, other constraints were needed to make solutions to his projection integrable (e.g., homogeneity, stationarity, periodicity, similarity). Recent extensions of these ideas show similarity considerations result in analytical solutions in non-homogeneous directions but for which the flow is of infinite extent.^{13–16} The problem with all of the efforts to apply this methodology to-date is the same as outlined above: any neglected dimension (especially if of infinite extent) smears out the phase information into the coefficients for the other directions. This complicates our understanding and makes actual physical interpretation impossible. The goal of this paper is to form an analysis of the two-spatial-point, two-time averaged equations through the use of a similarity solution.

II. THE BASIC EQUATIONS

All previous similarity analyses of decaying homogeneous turbulence begin with some form of the time-dependent two-spatial-point equations, either the two-point correlation functions, the structure functions, or their spectral counterparts.^{4,17,18} This analysis utilizes the two-spatial point, two-time equations. We have not seen these previously derived, but the derivation is straightforward as shown below.

As with the more classical two-spatial point equations, we begin with the equation for the fluctuating velocity, say u_i at point \mathbf{x} (which shall be represented with the Einsteinian notation $\mathbf{x} = x_i$ in the equations to follow) and time t , which in the absence of a mean flow reduces to

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} u_i u_j - \frac{\partial}{\partial x_j} \langle u_i u_j \rangle = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}. \quad (1)$$

The angled brackets $\langle \cdot \rangle$ denote an ensemble average. We have used the incompressible continuity equation to rewrite the second and third terms on the left-hand side, as well as simplifying the viscous term on the right-hand side for constant density and viscosity. We can write a similar equation at a different point,

say \mathbf{y} (also represented with Einsteinian notation $\mathbf{y} = y_i$), and denote the velocity at this point in space at a separate time, say \tilde{t} , as $\tilde{u}_k(\mathbf{y}, \tilde{t})$; i.e.,

$$\frac{\partial \tilde{u}_k}{\partial \tilde{t}} + \frac{\partial}{\partial y_j} \tilde{u}_k \tilde{u}_j - \frac{\partial}{\partial y_j} \langle \tilde{u}_k \tilde{u}_j \rangle = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial y_k} + \nu \frac{\partial^2 \tilde{u}_k}{\partial y_j \partial y_j}. \quad (2)$$

Note that \mathbf{y} represents a position to be evaluated at time \tilde{t} , while \mathbf{x} is to be evaluated at time t , as can be seen in Figure 1. It is important to distinguish this because the separation vector $\mathbf{r} = \mathbf{y}^{(a)} - \mathbf{x}^{(a)} = \mathbf{y}^{(b)} - \mathbf{x}^{(b)}$ is dependent on two times, not just one or the other. Note that superscripts refer to the different choice of origin, of which the correlation is hypothesized to be independent.

We can create two-point two-time (spatial and temporal) equations by multiplying the first equation by \tilde{u}_k , the second by u_i , adding, and averaging. The result is

$$\begin{aligned} & \frac{\partial}{\partial t} \langle u_i \tilde{u}_k \rangle + \frac{\partial}{\partial \tilde{t}} \langle u_i \tilde{u}_k \rangle + \frac{\partial}{\partial x_j} \langle u_i u_j \tilde{u}_k \rangle + \frac{\partial}{\partial y_j} \langle u_i \tilde{u}_j \tilde{u}_k \rangle \\ & = -\frac{1}{\rho} \frac{\partial \langle p \tilde{u}_k \rangle}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle u_i \tilde{p} \rangle}{\partial y_k} + \nu \frac{\partial^2 \langle u_i \tilde{u}_k \rangle}{\partial x_j \partial x_j} + \nu \frac{\partial^2 \langle u_i \tilde{u}_k \rangle}{\partial y_j \partial y_j}. \end{aligned} \quad (3)$$

Note that we have used the fact that u_i is not a function of \mathbf{y} or \tilde{t} , nor is \tilde{u}_k a function of \mathbf{x} or t , to allow us to move the respective variables into the derivatives.

It is proposed that homogeneity holds across time since the entire flow field is homogeneous at each time, but this will not be explicitly assumed in Section III. However, if homogeneity does hold, then the derivatives can be transformed into derivatives with respect to the separation vector $\mathbf{r}(t, \tilde{t}) = \mathbf{y}(\tilde{t}) - \mathbf{x}(t)$,

$$\frac{\partial}{\partial x_j} = -\frac{\partial}{\partial r_j} \quad \text{and} \quad \frac{\partial}{\partial y_j} = \frac{\partial}{\partial r_j} \quad (4)$$

and substitution of these derivatives into Equation (3) results in

$$\begin{aligned} & \frac{\partial}{\partial t} \langle u_i \tilde{u}_k \rangle + \frac{\partial}{\partial \tilde{t}} \langle u_i \tilde{u}_k \rangle + \frac{\partial}{\partial r_j} \left[\langle u_i \tilde{u}_j \tilde{u}_k \rangle - \langle u_i u_j \tilde{u}_k \rangle \right] \\ & = \frac{1}{\rho} \left[\frac{\partial}{\partial r_i} \langle p \tilde{u}_k \rangle - \frac{\partial}{\partial r_k} \langle u_i \tilde{p} \rangle \right] + 2\nu \frac{\partial^2}{\partial r_j^2} \langle u_i \tilde{u}_k \rangle. \end{aligned} \quad (5)$$

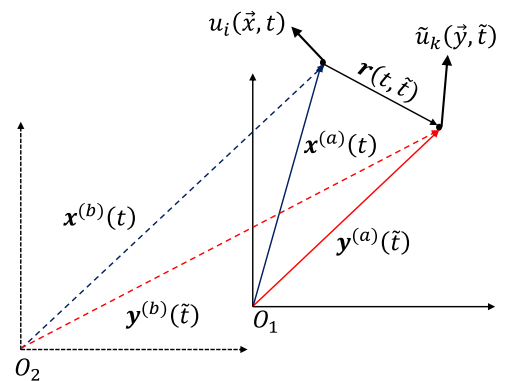


FIG. 1. Velocity u_i at time t and location \mathbf{x} compared to velocity \tilde{u}_k at time \tilde{t} and location \mathbf{y} . Points with a superscript of (b) correspond to the origin O_2 , while points with a superscript of (a) correspond to the origin O_1 . The relation between the two points is proposed to only depend on their separation $\mathbf{r}(t, \tilde{t})$, not the origin.

When $t = \tilde{t}$, Equation (5) reduces to the two-point single-time equation for homogeneous turbulence with no mean flow as shown in Monin and Yaglom,¹⁷

$$\begin{aligned} \frac{\partial}{\partial t} \langle u_i \tilde{u}_k \rangle + \frac{\partial}{\partial r_j} \left[\langle u_i \tilde{u}_j \tilde{u}_k \rangle - \langle u_i u_j \tilde{u}_k \rangle \right] \\ = \frac{1}{\rho} \left[\frac{\partial}{\partial r_i} \langle p \tilde{u}_k \rangle - \frac{\partial}{\partial r_k} \langle u_i \tilde{p} \rangle \right] + 2\nu \frac{\partial^2}{\partial r_j^2} \langle u_i \tilde{u}_k \rangle, \end{aligned} \quad (6)$$

where $\mathbf{r} = \mathbf{r}(t)$ only, since the second time is removed from the relation. To better understand how this occurs, refer to Figures 1 and 2, which show the relation between the two points. The two-space two-time equation had points \mathbf{x} and \mathbf{y} depending on times t and \tilde{t} , which results in two time dependencies in their correlation. However, when $t = \tilde{t}$, the previous position \mathbf{y} which corresponded to a different time can instead be seen as point \mathbf{x}' at the same time. Therefore, all \tilde{t} terms reduce to t . The isotropic version of Equation (6) further reduces to the Von Karman/Howarth equation.¹

The reason the two time derivatives in Equation (3) collapse to a single time derivative in Equation (6) without a pre-factor of 2 stems from their functional independence from the opposite coordinate. The first two terms in Equation (3) were created by multiplying Equation (1) by \tilde{u}_k and Equation (2) by u_i , adding, then averaging to obtain

$$\langle \tilde{u}_k \frac{\partial}{\partial t} u_i \rangle + \langle u_i \frac{\partial}{\partial t} \tilde{u}_k \rangle + \dots \quad (7)$$

The \tilde{u}_k term was then moved into the t derivative due to its independence from that coordinate, and the same was done with u_i and \tilde{t} . Setting $\tilde{t} = t$ simplifies the expression

$$\langle \tilde{u}_k \frac{\partial}{\partial t} u_i \rangle + \langle u_i \frac{\partial}{\partial t} \tilde{u}_k \rangle = \frac{\partial}{\partial t} \langle u_i \tilde{u}_k \rangle. \quad (8)$$

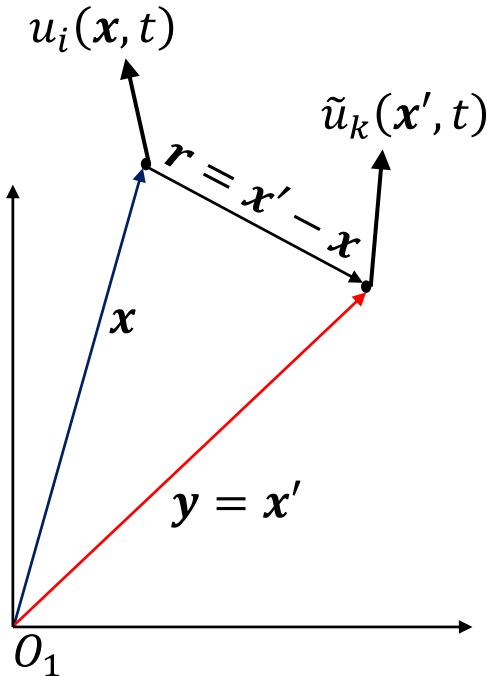


FIG. 2. When $\tilde{t} = t$, the relation between u_i and \tilde{u}_k reduces to the homogeneous two-point, single time results.

This is simply the chain rule, because both velocities now depend only on t .

III. THE TWO-POINT SPACE TIME SIMILARITY HYPOTHESIS

Since the two-point two-time equation for the two-point space-time correlation reduces to the two-point single time equation, then any two-time similarity solution must reduce to the two-point single time solution of George.⁸ What we expect is a solution which recognizes that the turbulence intensity decreases with time, and which also accommodates the change in spatial scale as all the measures of it increase with time. We will also take inspiration from Ewing *et al.*¹⁴ and expect that the similarity function will include some non-dimensional time coordinate that accommodates how time is “slowing” down as the scales get larger and the turbulence less intense.

It is hypothesized that the equations governing the two-point space, two-time velocity correlation tensor will admit to similarity solutions of the following type:

$$\langle u_i(\mathbf{x}, t) \tilde{u}_k(\mathbf{y}, \tilde{t}) \rangle = R_s(t, \tilde{t}) f_{i,k}(\boldsymbol{\eta}, \Theta), \quad (9)$$

where

$$\boldsymbol{\eta} = \frac{\mathbf{y} - \mathbf{x}}{\delta(t) + \delta'(\tilde{t})} = \frac{\mathbf{r}}{\delta + \delta'} \quad (10)$$

and Θ is some non-dimensional function of t and \tilde{t} , unknown *a priori*. The form of Equation (9) follows previous two-space, single time similarity solutions, with two new additions: the second time \tilde{t} is also included in the scaling parameter R_s , and a non-dimensional time Θ is included in the similarity function $f_{i,k}$. The inclusion of Θ is inspired by the work of Ewing *et al.*,¹⁴ where a dependence on the difference of the non-dimensional similarity coordinates was found. The length scales δ and δ' are to be dependent on their respective time only, but the homogeneous coordinate $\boldsymbol{\eta}$ is hypothesized to be scaled with the sum of them in order to retain scaling dependence on the two times. This form is assumed to allow the similarity coordinate to reduce to the two-space, single time similarity coordinate when $\tilde{t} = t$, under the informed assumption that the length scales will be the same functional form as was found by George.⁸

$R_s(t, \tilde{t})$, $\delta(t)$, and $\delta'(\tilde{t})$ are scale functions which are to be determined by insisting that all terms in the two-point space-time equations have the same time dependence, or none at all. This process is the same equilibrium similarity hypothesis used by George,⁸ where there is no assumption applied. Either the equations will admit to such solutions since all terms are in equilibrium similarity or they will not. (Note that a common reason for failure is that terms have been included in the analysis when they should have been neglected.)¹⁹

The pressure-velocity correlations and triple correlations also are hypothesized to have similarity solutions of a like form. Their inclusion into the following steps is omitted for the sake of brevity, but the full derivation is shown in the Appendix. Omission of these terms in the equations below does not influence the resulting solution.

We transform Equation (3) to spatial and temporal similarity coordinates, $\boldsymbol{\eta}$ and Θ , by substituting in the similarity

form of Equation (9) and appropriately applying the chain-rule. Note that this proposed solution is not being applied to the homogeneous form in Equation (5), but instead to the general form of Equation (3). If homogeneity does indeed apply across time, then the proposed similarity solution will satisfy Equation (3) and allow it to be written in the form of Equation (5). Plugging Equation (9) into Equation (3) gives

$$\begin{aligned} & \left[\frac{\partial R_s}{\partial t} \right] f_{i,k} - \left[\frac{R_s}{\delta + \delta'} \frac{d\delta}{dt} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} + \left[R_s \frac{\partial \Theta}{\partial t} \right] \frac{\partial f_{i,k}}{\partial \Theta} \\ & + \left[\frac{\partial R_s}{\partial \tilde{t}} \right] f_{i,k} - \left[\frac{R_s}{\delta + \delta'} \frac{d\delta'}{d\tilde{t}} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} + \left[R_s \frac{\partial \Theta}{\partial \tilde{t}} \right] \frac{\partial f_{i,k}}{\partial \Theta} \\ & + \dots = \dots + \left[\frac{2\nu R_s}{(\delta + \delta')^2} \right] \frac{\partial^2}{\partial \eta_j^2} f_{i,k}, \end{aligned} \quad (11)$$

where the \dots indicate the pressure-velocity and triple correlation terms. Note that in this equation, all terms with an explicit dependence on t or \tilde{t} only are contained in the brackets, while the scaling functions contain non-dimensional dependencies only. Equilibrium similarity of Equation (3) can only exist if all the terms with square brackets in Equation (11) are proportional as they evolve in time, or if the ratio of the different terms depends at most on Θ , the non-dimensional similarity variable in time.^{14,15}

To determine if the proposed similarity solution satisfies the equilibrium similarity hypothesis, we divide through by νR_s , multiply by $(\delta + \delta')^2$, and rearrange the terms. Additionally, if R_s is a separable function of the form

$$R_s(t, \tilde{t}) = U_s(t) U'_s(\tilde{t}), \quad (12)$$

then Equation (11) can be expressed as Equation (13),

$$\begin{aligned} & \left\{ \left[\frac{\delta^2}{\nu U_s} \frac{\partial U_s}{\partial t} \right] + 2 \left(\frac{\delta'}{\delta} \right) \left[\frac{\delta^2}{\nu U_s} \frac{\partial U_s}{\partial t} \right] + \left(\frac{\delta'}{\delta} \right)^2 \left[\frac{\delta^2}{\nu U_s} \frac{\partial U_s}{\partial t} \right] + \left[\frac{\delta'^2}{\nu U'_s} \frac{\partial U'_s}{\partial \tilde{t}} \right] + 2 \left(\frac{\delta}{\delta'} \right) \left[\frac{\delta'^2}{\nu U'_s} \frac{\partial U'_s}{\partial \tilde{t}} \right] + \left(\frac{\delta}{\delta'} \right)^2 \left[\frac{\delta'^2}{\nu U'_s} \frac{\partial U'_s}{\partial \tilde{t}} \right] \right\} f_{i,k} \\ & + \left\{ \left[\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \right] + 2 \left(\frac{\delta'}{\delta} \right) \left[\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \right] + \left(\frac{\delta'}{\delta} \right)^2 \left[\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \right] + \left(\frac{\delta}{\delta'} \right)^2 \left[\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] + 2 \left(\frac{\delta}{\delta'} \right) \left[\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] + \left[\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] \right\} \frac{\partial f_{i,k}}{\partial \Theta} \\ & = \dots + \left\{ \left[\frac{\delta}{\nu} \frac{d\delta}{dt} \right] + \left(\frac{\delta'}{\delta} \right) \left[\frac{\delta}{\nu} \frac{d\delta}{dt} \right] + \left(\frac{\delta}{\delta'} \right) \left[\frac{\delta'}{\nu} \frac{d\delta'}{d\tilde{t}} \right] + \left[\frac{\delta'}{\nu} \frac{d\delta'}{d\tilde{t}} \right] \right\} \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} + [2] \frac{\partial^2}{\partial \eta_j^2} f_{i,k}, \end{aligned} \quad (13)$$

$$\begin{aligned} & \left[\frac{\delta^2}{\nu U_s} \frac{\partial U_s}{\partial t} \right] f_{i,k} + \left[\frac{\delta'^2}{\nu U'_s} \frac{\partial U'_s}{\partial \tilde{t}} \right] f_{i,k} + \left[\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \right] \frac{\partial f_{i,k}}{\partial \Theta} + \left[\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] \frac{\partial f_{i,k}}{\partial \Theta} - \left[\frac{\delta}{\nu} \frac{d\delta}{dt} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} - \left[\frac{\delta'}{\nu} \frac{d\delta'}{d\tilde{t}} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} - [2] \frac{\partial^2}{\partial \eta_j^2} f_{i,k} \\ & + \left(\frac{\delta'}{\delta} \right) \left\{ 2 \left[\frac{\delta^2}{\nu U_s} \frac{\partial U_s}{\partial t} \right] f_{i,k} + 2 \left[\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \right] \frac{\partial f_{i,k}}{\partial \Theta} - \left[\frac{\delta}{\nu} \frac{d\delta}{dt} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} \right\} + \left(\frac{\delta'}{\delta} \right)^2 \left\{ \left[\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \right] \frac{\partial f_{i,k}}{\partial \Theta} + \left[\frac{\delta^2}{\nu U_s} \frac{\partial U_s}{\partial t} \right] f_{i,k} \right\} + \dots \\ & + \left(\frac{\delta}{\delta'} \right) \left\{ 2 \left[\frac{\delta'^2}{\nu U'_s} \frac{\partial U'_s}{\partial \tilde{t}} \right] f_{i,k} + 2 \left[\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] \frac{\partial f_{i,k}}{\partial \Theta} - \left[\frac{\delta'}{\nu} \frac{d\delta'}{d\tilde{t}} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} \right\} + \left(\frac{\delta}{\delta'} \right)^2 \left\{ \left[\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] \frac{\partial f_{i,k}}{\partial \Theta} + \left[\frac{\delta'^2}{\nu U'_s} \frac{\partial U'_s}{\partial \tilde{t}} \right] f_{i,k} \right\} = 0. \end{aligned} \quad (14)$$

The last term in Equation (13) has a bracketed value that is temporally independent, implying that it and the square bracketed expressions in Equation (13) must have no dependence on t or \tilde{t} . This can be shown by comparing the prefactors of terms 1, 4, 7, 12, 13, 16, and 17 of Equation (13),

$$\begin{aligned} \frac{\delta^2}{\nu U_s} \frac{\partial U_s}{\partial t} & \sim \frac{\delta}{\nu} \frac{d\delta}{dt} \sim \frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \sim \frac{\delta'^2}{\nu U'_s} \frac{\partial U'_s}{\partial \tilde{t}} \\ & \sim \frac{\delta'}{\nu} \frac{d\delta'}{d\tilde{t}} \sim \frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \sim 2, \end{aligned} \quad (15)$$

where the \sim denotes the same time dependency. Each grouping of scaling parameters in Equation (15) is a function of either t or \tilde{t} only. This requires that $\partial \Theta / \partial \tilde{t}$ is not dependent on t and that $\partial \Theta / \partial t$ is not dependent on \tilde{t} , which will be shown to be a valid assumption in Section IV. Since each grouping has the same temporal dependency, the last term being a constant requires that these expressions are all constants.

This argument of temporal independence can be better visualized if Equation (13) is rearranged into Equation (14),

where the first line has all the time independent groupings. The second and third lines of Equation (14) have the exact same terms as the first line, but with pre-factors of length scale ratios. These ratios must at most be functions of Θ , which is shown below, otherwise they will violate the equilibrium similarity hypothesis.

Another way to represent this argument is to use the method shown by both Ewing *et al.*¹⁴ and Wänström,¹⁵ where the ratio of any two separate scaling prefactors can be shown to be at most a function of the similarity coordinate Θ . For example, the ratio of the prefactors of terms 1 and 4 of Equation (11) is

$$\left[U'_s \frac{\partial U_s}{\partial t} \right] / \left[U_s \frac{\partial U'_s}{\partial \tilde{t}} \right]. \quad (16)$$

Utilizing the constraints of Equation (15) allows Equation (16) to be re-written as $(\delta'/\delta)^2$, to within a constant. Since each length scale is proposed to be a function of one time only, δ'^2/δ^2 cannot always be time independent for all t or \tilde{t} and

must then be a function of Θ ,

$$\left[U'_s \frac{\partial U_s}{\partial t} \right] / \left[U_s \frac{\partial U'_s}{\partial \tilde{t}} \right] \sim \left(\frac{\delta'}{\delta} \right)^2 = \Gamma(\Theta). \quad (17)$$

This ratio of length scales is exactly one of the groupings that show up in Equation (14). The other ratios of length scales that are in Equations (13) and (14), namely $(\delta/\delta')^2$, (δ/δ') , and (δ'/δ) , are just modified functions of Equation (17). As has been argued, these length scale ratios must be shown to be a function of Θ only in order for equilibrium similarity to hold, which will be shown in Section IV.

From these considerations, we conclude that similarity solutions are only possible if

$$\left[\frac{\delta}{\nu} \frac{d\delta}{dt} \right] = \left[\frac{\delta'}{\nu} \frac{d\delta'}{d\tilde{t}} \right] = A(\star), \quad (18)$$

$$\left[\frac{\delta^2}{\nu U_s} \frac{dU_s}{dt} \right] = \left[\frac{\delta'^2}{\nu U'_s} \frac{dU'_s}{d\tilde{t}} \right] = B(\star), \quad (19)$$

$$\left[\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \right] = C_1(\star), \quad (20)$$

$$\left[\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] = C_2(\star), \quad (21)$$

where $A(\star)$, $B(\star)$, $C_1(\star)$, and $C_2(\star)$ are constants that are at most functions of initial conditions (indicated by \star), and the equality across Equations (18) and (19) holds since the solutions must be the same when $t = \tilde{t}$. The reason Equations (20) and (21) have different constants is due to this condition not existing for the single time solution, so they may not necessarily have the same constant factor. Equation (18) can readily be solved,

$$\delta^2 = \delta_o^2 + 2A \nu(t - t_o), \quad (22)$$

$$\delta'^2 = \delta_o'^2 + 2A \nu(\tilde{t} - t_o), \quad (23)$$

where we have chosen the initial condition and time origin to be the same for both. These can be substituted into Equation (19) to obtain

$$\left[\frac{\delta_o^2 + 2A \nu(t - t_o)}{\nu U_s} \frac{dU_s}{dt} \right] = B, \quad (24)$$

which can in turn be rewritten as

$$\frac{d \ln U_s}{d \ln[\delta_o^2 + 2A \nu(t - t_o)]} = \frac{B}{2A}, \quad (25)$$

or more simply

$$\frac{d \ln U_s}{d \ln(\delta^2)} = \frac{B}{2A}. \quad (26)$$

An analogous expression holds for U'_s and δ' .

Integration from t_o to t yields the following solution for U_s :

$$\ln \left\{ \frac{U_s(t)}{U_s(t_o)} \right\} = \frac{B}{2A} \ln \left\{ \frac{\delta^2(t)}{\delta_o^2} \right\} \quad (27)$$

$$= \frac{B}{2A} \ln \left\{ 1 + \frac{2A\nu}{\delta_o^2} (t - t_o) \right\}. \quad (28)$$

This form of Equation (28) will be seen to have interesting implications for both long and short time decay. Explicitly solving for U_s gives

$$\frac{U_s(t)}{U_{s,o}} = \left\{ \frac{\delta^2(t)}{\delta_o^2} \right\}^{B/2A} = \left\{ 1 + \frac{2A\nu}{\delta_o^2} (t - t_o) \right\}^{B/2A}. \quad (29)$$

Similarly, U'_s is given by

$$\frac{U'_s(\tilde{t})}{U_{s,o}} = \left\{ \frac{\delta'^2(\tilde{t})}{\delta_o'^2} \right\}^{B/2A} = \left\{ 1 + \frac{2A\nu}{\delta_o'^2} (\tilde{t} - t_o) \right\}^{B/2A}. \quad (30)$$

Then $R_s(t, \tilde{t})$ is given by the product

$$R_s(t, \tilde{t}) = U_{s,o}^2 \left\{ \frac{\delta^2(t) \delta'^2(\tilde{t})}{\delta_o^4} \right\}^{B/2A} \quad (31)$$

$$= U_{s,o}^2 \left\{ \left[1 + \frac{2A\nu}{\delta_o^2} (t - t_o) \right] \left[1 + \frac{2A\nu}{\delta_o'^2} (\tilde{t} - t_o) \right] \right\}^{B/2A}. \quad (32)$$

So the final two-point, two-time correlation reduces to simply

$$\langle u_i \tilde{u}_k \rangle = U_{s,o}^2 \left\{ \frac{\delta^2 \delta'^2}{\delta_o^4} \right\}^{B/2A} f_{i,k}(\boldsymbol{\eta}, \Theta) \quad (33)$$

with the braced term equal to the braced term in Equation (32), and $B/2A$ dependent on the initial conditions.

IV. STATIONARITY IN THE STRETCHED TIME COORDINATE

With the scaling for δ known, the unknown function $\Theta(t, \tilde{t})$ can be found. Equations (20) and (21) imply

$$\left. \frac{\partial \Theta}{\partial t} \right|_{\tilde{t}=\text{const.}} = C_1(\star) \frac{\nu}{\delta^2}, \quad (34)$$

$$\left. \frac{\partial \Theta}{\partial \tilde{t}} \right|_{t=\text{const.}} = C_2(\star) \frac{\nu}{\delta'^2}. \quad (35)$$

The terms C_1 and C_2 are constants which can at most depend on the initial conditions. The results in Equations (22) and (23) directly imply that the derivatives with respect to time are related to the length scale (as shown in Equations (24)–(26)), and thus $\Theta(t, \tilde{t})$ can be written as follows:

$$\Theta = \frac{C_1}{2A} \ln(\delta^2) + \gamma_1(\tilde{t}) \quad \text{and} \quad \Theta = \frac{C_2}{2A} \ln(\delta'^2) + \gamma_2(t),$$

where γ_1 and γ_2 are functions of \tilde{t} and t , respectively. These results can immediately be combined

$$\Theta = \frac{C_1}{2A} \ln(\delta^2) + \frac{C_2}{2A} \ln(\delta'^2) + \gamma_3, \quad (36)$$

where γ_3 is a constant. Since any solution to the two-point two-time equation must reduce to the two-point single-time equation when $t = \tilde{t}$, the constants C_1 and C_2 can be constrained. The scaling function $f_{i,k}$ is not dependent on time in the two-point single-time similarity solution, so $f_{i,k}(\boldsymbol{\eta}, \Theta) \rightarrow f_{i,k}(\boldsymbol{\eta})$ for all $t = \tilde{t}$, implying that Θ is a constant or zero. This can be done for all $t = \tilde{t}$ if $C_1 = -C_2$.

To find these constants, we look to the fully transformed equation of motion. As was argued in Section III,

equilibrium similarity requires that the ratio of any bracketed terms in Equation (11) be either constant or a function of the similarity coordinate $\Theta(t, \tilde{t})$, since all time dependence is contained in these terms.¹⁴ This can easily be shown by Equation (16), which indicates the ratio of similarity functions is a constant times δ'^2/δ^2 , requiring

$$\frac{\delta'^2}{\delta^2} = \Gamma(\Theta) \quad (37)$$

for some function Γ of the similarity coordinate. This can equivalently be written as follows:

$$\ln(\delta'^2) - \ln(\delta^2) = \ln(\Gamma(\Theta)) = \Phi(\Theta) \quad (38)$$

for some different function Φ of Θ . Therefore, we can set $C_1 = -C_2 = -1$ and $\gamma_3 = 0$ to get the similarity coordinate for time as follows:

$$\Theta = \frac{1}{2A} \ln \left(\frac{\delta_0^2 + 2A\nu(\tilde{t} - t_o)}{\delta_0^2 + 2A\nu(t - t_o)} \right). \quad (39)$$

Substituting the relation for Θ into Equation (11) and comparing terms 3 and 6 confirms the result for Θ ,

$$\begin{aligned} \dots + \left[R_s \frac{\partial \Theta}{\partial t} \right] \frac{\partial f_{i,k}}{\partial \Theta} + \left[R_s \frac{\partial \Theta}{\partial \tilde{t}} \right] \frac{\partial f_{i,k}}{\partial \Theta} + \dots \\ \text{or} \\ \dots - \left[\frac{\nu R_s}{\delta^2} \right] \frac{\partial f_{i,k}}{\partial \Theta} + \left[\frac{\nu R_s}{\delta'^2} \right] \frac{\partial f_{i,k}}{\partial \Theta} + \dots \end{aligned}$$

When $t = \tilde{t}$, then $\delta^2 = \delta'^2$, canceling the two terms resulting in all gradients in Θ being removed. Since the equation of motion then has no derivatives with respect to Θ , and with $\Theta = 0$ for

all time, the scaling function $f_{i,k}$ does not change with time and the single time result is recovered.

With Θ defined, we can now write the ratios of length scales as a function of Θ ,

$$\left(\frac{\delta'}{\delta} \right)^2 = e^{2A\Theta}, \quad (40)$$

$$\frac{\delta'}{\delta} = e^{A\Theta}. \quad (41)$$

The result for Θ means that the similarity function $f_{i,k}$ is dependent on the difference in the logarithms of time. In other words, the absolute time does not determine the two-space two-time correlation, only the difference in log times. This temporal dependence can be thought of as a “stretching” in time, since larger physical time differences are needed at longer times to produce the same logarithmic difference. Therefore, it becomes apparent that this scaling shows that the scaled statistics are stationary in log time.

The basic phenomenon being scaled out here is the fact that as the turbulence decays, the “eddies” are slowing down, but when time is measured in logarithmic increments, it is not. All logarithmic time differences appear to be the same, no matter where in the decay life cycle the turbulence finds itself.

V. FULL TWO-POINT, TWO-TIME SIMILARITY EQUATION

Substituting in the results of the similarity functions of Equations (22), (23), (29), (30), (39), and (A28)–(A31) into the full equation of (A14c) gives the full two-point, two-time equation in similarity coordinates,

$$\begin{aligned} B \left\{ \left(1 + e^{A\Theta} \right)^2 + \left(1 + e^{-A\Theta} \right)^2 \right\} f_{i,k} - A \left(2 + e^{A\Theta} + e^{-A\Theta} \right) \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} + \left\{ \left(1 + e^{-A\Theta} \right)^2 - \left(1 + e^{A\Theta} \right)^2 \right\} \frac{\partial f_{i,k}}{\partial \Theta} \\ - D \left(1 + e^{A\Theta} \right) \frac{\partial t_{ij,k}}{\partial \eta_j} + D \left(1 + e^{-A\Theta} \right) \frac{\partial t_{i,jk}}{\partial \eta_j} = E \left(1 + e^{A\Theta} \right) \frac{\partial p_{ak}}{\partial \eta_i} - E \left(1 + e^{-A\Theta} \right) \frac{\partial p_{bi}}{\partial \eta_k} + 2 \frac{\partial^2}{\partial \eta_j^2} f_{i,k}, \end{aligned} \quad (42)$$

where A , B , D , and E are all constants that depend on initial conditions. If $t = \tilde{t}$, then $\Theta = 0$ and the equation loses all dependence on it, resulting in the single time equation. Note that all terms in Equation (42) are functions of the similarity coordinates $\boldsymbol{\eta}$ or Θ , showing that the proposed scaling and similarity variables yield a similarity solution that is stationary in log time and homogeneous in space.

VI. RELATION TO TWO-POINT, SINGLE-TIME RESULT

If $t = \tilde{t}$, Equation (33) simplifies to the two-point single-time correlation,

$$\langle u_i \tilde{u}_k \rangle = U_{s,o}^2 \left\{ 1 + (2A\nu(t - t_o)/\delta_o^2) \right\}^{B/A} f_{i,k}(\boldsymbol{\eta}). \quad (43)$$

This can further be simplified by looking at the trace for zero spatial separation between points, i.e., the TKE. Setting $i = k$ and $\boldsymbol{\eta} = 0$ results in

$$\langle u_i u_i \rangle = U_{s,o}^2 \left\{ 1 + (2A\nu(t - t_o)/\delta_o^2) \right\}^{B/A}. \quad (44)$$

Equation (44) is exactly the same result of George⁸ by seeing $B/A = n$, the energy decay rate (presumed negative). Either the Taylor microscale or the integral scale is the appropriate choice for the length scale. Since the Taylor microscale is most reliably measured, we take $\delta(t) = \lambda(t)$. With the Taylor microscale growing in time, the TKE will require a negative exponent to stay bounded in time, meaning $B/A < 0$. The functional form of the length scale implies that A must remain positive, meaning B is a negative number.

It is possible to use Equations (28) and (44) to show that both a power-law and exponential decay behavior are present, the former for large times, and the latter for short times. For intermediate times, the decay is more complicated. To see the short time behavior simply choose t to be small enough that

$2Av(t - t_o)/\lambda_o^2 \ll \lambda_o^2$. Expanding the logarithm in Equation (28) allows us to write

$$\ln[U_s/U_o] = Bv(t - t_o)/\lambda_o^2 + \dots \quad (45)$$

This immediately yields the exponential decay solution of George and Wang,²⁰ i.e.,

$$\langle u_i u_i \rangle = U_o^2 \exp[2Bv(t - t_o)/\lambda_o^2]. \quad (46)$$

George and Wang noted that for isotropic turbulence, the value of the numerical coefficient in the exponent was calculated to be -10 compared to our unknown $2B$ in a non-isotropic homogeneous flow. Note that this implies that B must be negative, which matches the previous conclusion about its sign.

It seems likely that ALL attempts to produce decaying turbulence will behave this way for short times, since all start with some finite value of λ_o . Therefore, it is probably worthwhile to re-examine old homogeneous turbulence data^{2,3} in light of this. If the exponential part is not included, then this could very much affect the virtual origin needed to fit a power law to the data, or the power exponent needed. Clearly the fractal grids accentuate this part of the solution more than the standard grids.²¹⁻²⁴

Now look at the large time behavior where $2Av(t - t_o)/\lambda_o^2 \gg \lambda_o^2$. In this limit,

$$\ln[U_s/U_o] \approx \ln [2Av(t - t_o)/\lambda_o^2]^{B/2A} \quad (47)$$

with the approximation holding better for larger time. This clearly yields a power law decay for large times; i.e.,

$$\langle u_i u_i \rangle = U_o^2 [2Av(t - t_o)/\lambda_o^2]^{B/A}. \quad (48)$$

Hence in the limit of large and small times, both the power law and exponential solutions are valid. Neither are exactly true for intermediate times, which may be for most experiments and simulations. Thus, Equation (44) is exactly the composite solution proposed by Mazellier and Vassilicos,²⁵ which combined the solutions of George⁸ and George and Wang²⁰ into a single formula. The role of the initial length scale, λ_o , does not seem to have been previously noted.

VII. DECAY RATE AND INITIAL CONDITIONS

We know from the results of Section VI, as well as from experimental results, that $\langle u^2 \rangle \propto t^n$ with $n \leq -1$. But stationarity in log-time variables implies the energy in time must be infinite, which is possible only if an infinite energy is added throughout space at the initial instant. This is consistent only with t^{-1} decay. But in no experiments or simulations has a t^{-1} decay ever been observed, and $n < -1$ is observed instead. This is exactly the problem we have with the finite spatial boundaries as well. Clearly further study is required to understand the role of finite boundaries and conditions on the flows we can actually realize. The two-space, two time similarity solution at least provides a clue as to what flow we are trying to achieve.

VIII. CONCLUSIONS AND FUTURE WORK

A similarity solution for the two-point, two-time averaged equations of motion for homogeneous turbulence in absence of a mean flow has been found. The similarity solution implies that the flow is homogeneous across time, as well as statistically stationary in logarithmic time increments. By utilizing these new scaled coordinates, it may be possible to decompose any homogeneous turbulent field in the absence of a mean flow into Fourier modes. The implications of this result are far reaching, including a new way to think about constructing a realization of a homogeneous decaying turbulence field, as well as reconsidering previous decompositions that neglected the transformation in the temporal coordinate. Future investigation of the proposed solution should include an attempt at solving a “dynamically scaled” DNS that utilizes a growing grid spacing and logarithmic time steps.

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APPENDIX: FORMAL DERIVATION OF THE SIMILARITY SOLUTION

The two-space two-time homogeneous turbulence equation was found through averaging the product of the single point fluctuating equations at two separate locations, \mathbf{x} and \mathbf{y} , and two separate times t and \tilde{t} . The equation is found by starting with Equations (1) and (2), and multiplying the first by \tilde{u}_k , and the second by u_i ,

$$\tilde{u}_k \frac{\partial u_i}{\partial t} + \tilde{u}_k \frac{\partial}{\partial x_j} u_i u_j - \tilde{u}_k \frac{\partial}{\partial x_j} \langle u_i u_j \rangle = -\tilde{u}_k \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \tilde{u}_k \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad (A1a)$$

$$u_i \frac{\partial \tilde{u}_k}{\partial \tilde{t}} + u_i \frac{\partial}{\partial y_j} \tilde{u}_k \tilde{u}_j - u_i \frac{\partial}{\partial y_j} \langle \tilde{u}_k \tilde{u}_j \rangle = -u_i \frac{1}{\rho} \frac{\partial \tilde{p}}{\partial y_k} + u_i \nu \frac{\partial^2 \tilde{u}_k}{\partial y_j \partial y_j}. \quad (A1b)$$

Next, the \tilde{u}_k can be moved inside the derivatives in Equation (A1a) because it is a function of \mathbf{y} and \tilde{t} , while the derivatives are with respect to \mathbf{x} and t . Likewise, the u_i can be moved inside the derivatives in Equation (A1b) because it is a function of \mathbf{x} and t , while the derivatives are with respect to \mathbf{y} and \tilde{t} . Performing an ensemble average on each equation then will have the third term disappear in both expressions, resulting in

$$\frac{\partial \langle u_i \tilde{u}_k \rangle}{\partial t} + \frac{\partial}{\partial x_j} \langle u_i u_j \tilde{u}_k \rangle = -\frac{1}{\rho} \frac{\partial \langle p \tilde{u}_k \rangle}{\partial x_i} + \nu \frac{\partial^2 \langle u_i \tilde{u}_k \rangle}{\partial x_j \partial x_j}, \quad (A2a)$$

$$\frac{\partial \langle u_i \tilde{u}_k \rangle}{\partial \tilde{t}} + \frac{\partial}{\partial y_j} \langle u_i \tilde{u}_k \tilde{u}_j \rangle = -\frac{1}{\rho} \frac{\partial \langle u_i \tilde{p} \rangle}{\partial y_k} + \nu \frac{\partial^2 \langle u_i \tilde{u}_k \rangle}{\partial y_j \partial y_j}. \quad (A2b)$$

Summation of Equations (A2a) and (A2b) then gives

$$\begin{aligned} & \frac{\partial}{\partial t} \langle u_i \tilde{u}_k \rangle + \frac{\partial}{\partial \tilde{t}} \langle u_i \tilde{u}_k \rangle + \frac{\partial}{\partial x_j} \langle u_i u_j \tilde{u}_k \rangle + \frac{\partial}{\partial y_j} \langle u_i \tilde{u}_j \tilde{u}_k \rangle \\ &= -\frac{1}{\rho} \frac{\partial \langle p \tilde{u}_k \rangle}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle u_i \tilde{p} \rangle}{\partial y_k} + \nu \frac{\partial^2 \langle u_i \tilde{u}_k \rangle}{\partial x_j \partial x_j} + \nu \frac{\partial^2 \langle u_i \tilde{u}_k \rangle}{\partial y_j \partial y_j}. \end{aligned} \quad (\text{A3})$$

We will start by assuming a similarity solution for the two-point, two-time velocity correlation of the following form:

$$\langle u_i(\mathbf{x}, t) \tilde{u}_k(\mathbf{y}, \tilde{t}) \rangle = R_s(t, \tilde{t}) f_{i,k}(\boldsymbol{\eta}, \Theta), \quad (\text{A4})$$

$$\langle u_i u_j \tilde{u}_k \rangle = T_a(t, \tilde{t}) t_{ij,k}(\boldsymbol{\eta}, \Theta), \quad (\text{A5})$$

$$\langle u_i \tilde{u}_j \tilde{u}_k \rangle = T_b(t, \tilde{t}) t_{i,jk}(\boldsymbol{\eta}, \Theta), \quad (\text{A6})$$

$$\langle p \tilde{u}_k \rangle = P_a(t, \tilde{t}) p_{ak}(\boldsymbol{\eta}, \Theta), \quad (\text{A7})$$

$$\langle \tilde{p} u_i \rangle = P_b(t, \tilde{t}) p_{bi}(\boldsymbol{\eta}, \Theta), \quad (\text{A8})$$

where $\boldsymbol{\eta} = (\mathbf{y} - \mathbf{x})/(\delta + \delta')$ and Θ is some non-dimensional function of t and \tilde{t} , unknown *a priori*. $R_s(t, \tilde{t})$, $\delta(t)$, and $\delta'(\tilde{t})$ are scale functions which are to be determined by insisting that all terms in the two-point space-time equations have the same time dependence, or none at all. This process is the same equilibrium similarity hypothesis used by George,⁸ where there is no *assumption* applied. Either the equations will admit to such solutions since all terms are in equilibrium similarity, or they will not. (Note that a common reason for failure is that terms have been included in the analysis when they should have been neglected.¹⁹)

Note that R_s , T_a , T_b , P_a , and P_b are scaling functions which contain all dimensions, while the corresponding similarity functions of $\boldsymbol{\eta}$ and Θ are non-dimensional.

It is important to note the difference in functions, where $t_{ij,k}(\boldsymbol{\eta}, \Theta) \neq t_{i,jk}(\boldsymbol{\eta}, \Theta)$ and $p_{ak}(\boldsymbol{\eta}, \Theta) \neq p_{bi}(\boldsymbol{\eta}, \Theta)$, even if $i = k$. Substituting these assumed solutions into Equation (A3) and carrying out the math, we will get a new set of time dependent constraints.

Looking to term 3 of Equation (A3), corresponding to $\frac{\partial}{\partial x_j} \langle u_i u_j \tilde{u}_k \rangle$, we will substitute the similarity form to find

$$\begin{aligned} \frac{\partial}{\partial x_j} \langle u_i u_j \tilde{u}_k \rangle &= \frac{\partial}{\partial x_j} \left(T_a(t, \tilde{t}) t_{ij,k}(\boldsymbol{\eta}, \Theta) \right) \\ &= T_a \frac{\partial t_{ij,k}}{\partial \eta_j} \frac{\partial \eta_j}{\partial x_j} \\ &= - \left[\frac{T_a}{\delta + \delta'} \right] \frac{\partial t_{ij,k}}{\partial \eta_j}. \end{aligned}$$

Likewise, substituting Equation (A4) into the first term of Equation (A3) will result in

$$\begin{aligned} \frac{\partial}{\partial t} \langle u_i \tilde{u}_k \rangle &= \frac{\partial}{\partial t} \left(R_s(t, \tilde{t}) f_{i,k}(\boldsymbol{\eta}, \Theta) \right) \\ &= \frac{\partial R_s}{\partial t} f_{i,k} + R_s \frac{\partial f_{i,k}}{\partial \eta_j} \frac{d \eta_j}{d t} \frac{d \delta}{d t} + R_s \frac{\partial f_{i,k}}{\partial \Theta} \frac{d \Theta}{d t} \\ &= \left[\frac{\partial R_s}{\partial t} \right] f_{i,k} - R_s \frac{r_j}{(\delta + \delta')^2} \frac{d \delta}{d t} \frac{\partial f_{i,k}}{\partial \eta_j} + \left[R_s \frac{d \Theta}{d t} \right] \frac{\partial f_{i,k}}{\partial \Theta} \\ &= \left[\frac{\partial R_s}{\partial t} \right] f_{i,k} - \left[\frac{R_s}{\delta + \delta'} \frac{d \delta}{d t} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} + \left[R_s \frac{d \Theta}{d t} \right] \frac{\partial f_{i,k}}{\partial \Theta}. \end{aligned}$$

The same process applied to all of the other terms, and taking care to use the chain rule, will result in Equation (A14a) for the full similarity equation. Note that in this equation, all terms with an explicit dependence on t or \tilde{t} are contained in the square brackets, while the scaling functions contain only non-dimensional dependencies. Equilibrium similarity of Equation (A14a) can only exist if all the terms with square brackets are proportional as they evolve in time, or if the ratio of the different terms at most depends on Θ ,^{14,15} the non-dimensional similarity variable in time.

To determine if the proposed similarity solution satisfies the equilibrium similarity hypothesis, we divide through by νR_s and multiply by $(\delta + \delta')^2$, giving us Equation (A14b). This new expression is not simplified enough to inform us of the temporal dependency of the scaling parameters. We next take inspiration from the single time results, where in order to determine the functional form of the scaling parameters we assume separability,

$$R_s(t, \tilde{t}) = U_s(t) U'_s(\tilde{t}), \quad (\text{A9})$$

$$T_a(t, \tilde{t}) = K_t(t) U'_s(\tilde{t}), \quad (\text{A10})$$

$$T_b(t, \tilde{t}) = K'_t(\tilde{t}) U_s(t), \quad (\text{A11})$$

$$P_a(t, \tilde{t}) = P_s(t) U'_s(\tilde{t}), \quad (\text{A12})$$

$$P_b(t, \tilde{t}) = P'_s(\tilde{t}) U_s(t). \quad (\text{A13})$$

This allows a t or \tilde{t} dependency in each square bracketed term to cancel out, ensuring that the scaling parameters within each bracketed term are dependent on only t or only \tilde{t} . We will treat the function Θ shortly, but for now assume that its gradient is a function of t or \tilde{t} when appropriate. Plugging Equations (A9)–(A13) into Equation (A14b), along with further re-arrangement, then results in Equation (A14c),

$$\begin{aligned} & \left[\frac{\partial R_s}{\partial t} \right] f_{i,k} - \left[\frac{R_s}{\delta + \delta'} \frac{d \delta}{d t} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} + \left[R_s \frac{\partial \Theta}{\partial t} \right] \frac{\partial f_{i,k}}{\partial \Theta} + \left[\frac{\partial R_s}{\partial \tilde{t}} \right] f_{i,k} - \left[\frac{R_s}{\delta + \delta'} \frac{d \delta'}{d \tilde{t}} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} + \left[R_s \frac{\partial \Theta}{\partial \tilde{t}} \right] \frac{\partial f_{i,k}}{\partial \Theta} \\ & - \left[\frac{T_a}{\delta + \delta'} \right] \frac{\partial t_{ij,k}}{\partial \eta_j} + \left[\frac{T_b}{\delta + \delta'} \right] \frac{\partial t_{i,jk}}{\partial \eta_j} = \left[\frac{P_a}{\rho(\delta + \delta')} \right] \frac{\partial p_{ak}}{\partial \eta_i} - \left[\frac{P_b}{\rho(\delta + \delta')} \right] \frac{\partial p_{bi}}{\partial \eta_k} + \left[\frac{2\nu R_s}{(\delta + \delta')^2} \right] \frac{\partial^2 f_{i,k}}{\partial \eta_j^2}, \end{aligned} \quad (\text{A14a})$$

$$\begin{aligned}
& \left[\frac{(\delta + \delta')^2}{\nu R_s} \frac{\partial R_s}{\partial t} \right] f_{i,k} - \left[\frac{(\delta + \delta')}{\nu} \frac{d\delta}{dt} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} + \left[\frac{(\delta + \delta')^2}{\nu} \frac{\partial \Theta}{\partial t} \right] \frac{\partial f_{i,k}}{\partial \Theta} + \left[\frac{(\delta + \delta')^2}{\nu R_s} \frac{\partial R_s}{\partial \tilde{t}} \right] f_{i,k} \\
& - \left[\frac{(\delta + \delta')}{\nu} \frac{d\delta'}{d\tilde{t}} \right] \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} + \left[\frac{(\delta + \delta')^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] \frac{\partial f_{i,k}}{\partial \Theta} - \left[\frac{(\delta + \delta')}{\nu} \frac{T_a}{R_s} \right] \frac{\partial t_{ij,k}}{\partial \eta_j} \\
& + \left[\frac{(\delta + \delta')}{\nu} \frac{T_b}{R_s} \right] \frac{\partial t_{ijk}}{\partial \eta_j} = \left[\frac{(\delta + \delta')}{\rho \nu} \frac{P_a}{R_s} \right] \frac{\partial p_{ak}}{\partial \eta_i} - \left[\frac{(\delta + \delta')}{\rho \nu} \frac{P_b}{R_s} \right] \frac{\partial p_{bi}}{\partial \eta_k} + [2] \frac{\partial^2}{\partial \eta_j^2} f_{i,k}, \quad (\text{A14b})
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left[\frac{\delta^2}{\nu U_s} \frac{\partial U_s}{\partial t} \right] + 2 \left(\frac{\delta'}{\delta} \right) \left[\frac{\delta^2}{\nu U_s} \frac{\partial U_s}{\partial t} \right] + \left(\frac{\delta'}{\delta} \right)^2 \left[\frac{\delta^2}{\nu U_s} \frac{\partial U_s}{\partial t} \right] + \left[\frac{\delta'^2}{\nu U'_s} \frac{\partial U'_s}{\partial \tilde{t}} \right] + 2 \left(\frac{\delta}{\delta'} \right) \left[\frac{\delta'^2}{\nu U'_s} \frac{\partial U'_s}{\partial \tilde{t}} \right] + \left(\frac{\delta}{\delta'} \right)^2 \left[\frac{\delta'^2}{\nu U'_s} \frac{\partial U'_s}{\partial \tilde{t}} \right] \right\} f_{i,k} \\
& - \left\{ \left[\frac{\delta}{\nu} \frac{d\delta}{dt} \right] + \left(\frac{\delta'}{\delta} \right) \left[\frac{\delta}{\nu} \frac{d\delta}{dt} \right] + \left(\frac{\delta}{\delta'} \right) \left[\frac{\delta'}{\nu} \frac{d\delta'}{d\tilde{t}} \right] + \left[\frac{\delta'}{\nu} \frac{d\delta'}{d\tilde{t}} \right] \right\} \eta_j \frac{\partial f_{i,k}}{\partial \eta_j} \\
& + \left\{ \left[\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \right] + 2 \left(\frac{\delta'}{\delta} \right) \left[\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \right] + \left(\frac{\delta'}{\delta} \right)^2 \left[\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \right] + \left(\frac{\delta}{\delta'} \right)^2 \left[\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] + 2 \left(\frac{\delta}{\delta'} \right) \left[\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] + \left[\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \right] \right\} \frac{\partial f_{i,k}}{\partial \Theta} \\
& - \left\{ \left[\frac{\delta}{\nu} \frac{K_t}{U_s} \right] + \left(\frac{\delta'}{\delta} \right) \left[\frac{\delta}{\nu} \frac{K_t}{U_s} \right] \right\} \frac{\partial t_{ij,k}}{\partial \eta_j} + \left\{ \left(\frac{\delta}{\delta'} \right) \left[\frac{\delta'}{\nu} \frac{K'_t}{U'_s} \right] + \left[\frac{\delta'}{\nu} \frac{K'_t}{U'_s} \right] \right\} \frac{\partial t_{ijk}}{\partial \eta_j} = \left\{ \left[\frac{\delta}{\rho \nu} \frac{P_s}{U_s} \right] + \left(\frac{\delta'}{\delta} \right) \left[\frac{\delta}{\rho \nu} \frac{P_s}{U_s} \right] \right\} \frac{\partial p_{ak}}{\partial \eta_i} \\
& - \left\{ \left(\frac{\delta}{\delta'} \right) \left[\frac{\delta'}{\rho \nu} \frac{P'_s}{U'_s} \right] + \left[\frac{\delta'}{\rho \nu} \frac{P'_s}{U'_s} \right] \right\} \frac{\partial p_{bi}}{\partial \eta_k} + [2] \frac{\partial^2}{\partial \eta_j^2} f_{i,k}. \quad (\text{A14c})
\end{aligned}$$

Careful examination of the time dependencies of the five proposed similarity functions in Equations (A9)–(A13) shows that the separability assumption yields a valid solution. For example, looking at the third term in Equation (A3), corresponding to $\frac{\partial}{\partial x_j} \langle u_i u_j \tilde{u}_k \rangle$, it is apparent that the dependence on \tilde{t} lies only on one velocity variable, while the dependence on t lies in two velocity terms. With these dependencies on t and \tilde{t} , we have proposed that it can be separated into two components, and that the single velocity component is the same separable U'_s utilized in the expression $R_s = U_s(t)U'_s(\tilde{t})$. The same logic is applied to all the triple correlation and pressure-velocity correlation terms.

Inspection of terms contained in the braces of Equation (A14c) reveals that the parameters in square brackets must be temporally independent due to the constant factor multiplying the second derivative. In other words

$$\begin{aligned}
\frac{\delta^2}{\nu U_s} \frac{dU_s}{dt} & \sim \frac{\delta'^2}{\nu U'_s} \frac{dU'_s}{d\tilde{t}} \sim \frac{\delta}{\nu} \frac{d\delta}{dt} \sim \frac{\delta'}{\nu} \frac{d\delta'}{d\tilde{t}} \sim \frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} \\
& \sim \frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} \sim \frac{K_t \delta}{\nu U_s} \sim \frac{K'_t \delta'}{\nu U'_s} \sim \frac{P_s \delta}{\rho \nu U_s} \sim \frac{P'_s \delta'}{\rho \nu U'_s} \sim 2, \quad (\text{A15})
\end{aligned}$$

where \sim signifies that the terms have the same temporal dependence. Since the last value is a constant, all the terms in Equation (A15) are independent of t and \tilde{t} . This condition requires that $\partial \Theta / \partial t$ does not depend on \tilde{t} , and that $\partial \Theta / \partial \tilde{t}$ does not depend on t , but this assumption was shown to be valid in Section IV.

Finally, the ratio of any two groupings in the square brackets of Equation (A14a) must be at most a function of Θ . In other words, all the terms in parentheses in Equation (A14c) have to be a function of Θ , or a constant. If $t = \tilde{t}$, the length scales δ and δ' will be equal, implying that all ratios of them are 1. However, since the correlation takes place across all times t

and \tilde{t} , we cannot say in general that $\delta = \delta'$, which means they must be a function of Θ for equilibrium similarity to hold.

Under this constraint, we prescribe the following functions:

$$\left(\frac{\delta'}{\delta} \right)^2 = \Gamma_1(\Theta), \quad (\text{A16})$$

$$\frac{\delta'}{\delta} = \Gamma_2(\Theta). \quad (\text{A17})$$

If functions for Γ_1 and Γ_2 cannot be found, then equilibrium similarity does not hold for Equation (A14a), but the result of Section IV confirms the existence of a solution of the proposed form.

Given all these requirements, we conclude that similarity solutions are only possible if

$$\frac{\delta}{\nu} \frac{d\delta}{dt} = \frac{\delta'}{\nu} \frac{d\delta'}{d\tilde{t}} = A(\star), \quad (\text{A18})$$

$$\frac{\delta^2}{\nu U_s} \frac{dU_s}{dt} = \frac{\delta'^2}{\nu U'_s} \frac{dU'_s}{d\tilde{t}} = B(\star), \quad (\text{A19})$$

$$\frac{\delta^2}{\nu} \frac{\partial \Theta}{\partial t} = C_1(\star), \quad (\text{A20})$$

$$\frac{\delta'^2}{\nu} \frac{\partial \Theta}{\partial \tilde{t}} = C_2(\star), \quad (\text{A21})$$

$$\frac{K_t \delta}{\nu U_s} = \frac{K'_t \delta'}{\nu U'_s} = D(\star), \quad (\text{A22})$$

$$\frac{P_s \delta}{\rho \nu U_s} = \frac{P'_s \delta'}{\rho \nu U'_s} = E(\star). \quad (\text{A23})$$

Integration of Equation (A18) gives

$$\delta^2 = \delta_0^2 + 2A\nu(t - t_0), \quad (\text{A24})$$

$$\delta'^2 = \delta_0^2 + 2Av(\tilde{t} - t_0), \quad (\text{A25})$$

where it is implied that the initial conditions and initial length/time origin are the same for both expressions. Integration of Equation (A19) will yield

$$\ln\left(\frac{U_s}{U_{s,0}}\right) = \frac{B}{2A} \ln\left(\frac{\delta_0^2 + 2Av(t - t_0)}{\delta_0^2}\right)$$

or equivalently

$$U_s = U_{s,0} \left(1 + \frac{2Av}{\delta_0^2}(t - t_0)\right)^{B/2A}, \quad (\text{A26})$$

$$U'_s = U_{s,0} \left(1 + \frac{2Av}{\delta_0^2}(\tilde{t} - t_0)\right)^{B/2A}. \quad (\text{A27})$$

This too implies the same initial conditions and velocity scale for the second time \tilde{t} .

With the length and velocity scales found in Equations (A24)–(A27), the pressure and triple correlations scales can also be found

$$K_t = \frac{DvU_{s,0}}{\delta_0} \left[1 + \frac{2vA}{\delta_0^2}(t - t_0)\right]^{(B/A-1)/2}, \quad (\text{A28})$$

$$K'_t = \frac{DvU_{s,0}}{\delta_0} \left[1 + \frac{2vA}{\delta_0^2}(\tilde{t} - t_0)\right]^{(B/A-1)/2}, \quad (\text{A29})$$

and

$$\frac{P_s}{\rho} = \frac{EvU_{s,0}}{\delta_0} \left[1 + \frac{2vA}{\delta_0^2}(t - t_0)\right]^{(B/A-1)/2}, \quad (\text{A30})$$

$$\frac{P'_s}{\rho} = \frac{EvU_{s,0}}{\delta_0} \left[1 + \frac{2vA}{\delta_0^2}(\tilde{t} - t_0)\right]^{(B/A-1)/2}. \quad (\text{A31})$$

This new equation could then be used to determine, for example, the scaling of the triple correlation terms in Equation (A3),

$$\langle u_i u_j \tilde{u}_k \rangle = \frac{BDvU_{s,0}^2}{\delta_0} \left(\frac{\delta(t)}{\delta_0}\right)^{B/A-1} \left(\frac{\delta'(\tilde{t})}{\delta_0}\right)^{B/A} \cdot t_{ij,k}(\boldsymbol{\eta}, \boldsymbol{\Theta}).$$

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